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Rational interpolation method for solving initial value problems (IVPs) in ordinary differential equations

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Abstract

In this paper we designed Rational Interpolation Method for solving Ordinary Differential Equations (ODES) and Stiff initial value problems (IVPs).

This was achieved by considering the Rational Interpolation Formula.

$$y_{(x)} = U_{(x)} = \frac{P_0 + P_1 x + P_2 x^2 + P^3 x^3 + p_4 x^4 + P_5 x^5}{1 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4 + q_5 x^5 + q_6 x^6},$$

satisfying $U(X_{n+i}) = y_{n+i}$, $i = 0, 1, 2, 3, 4, 5, 6$.

We also implemented $k = 6$ in Aashikpelokhai (1991) class of rational integration formulas given by

$$y_{n+1} = \frac{\sum_{i=0}^{5} p_i X_{n+1}^i}{1 + \sum_{i=1}^{6} q_i X_{n+1}^i}$$

where,

$$p_i = \frac{(2k-1-i)! \binom{k-1}{i} \overline{h}^i}{(2k-1)! x_{n+1}^i} y_n, \quad i = 0(1)(k-1), \qquad q_i = \frac{(-1)^i \, (2k-1-i)! \binom{k}{i} \overline{h}^i}{(2k-1)! x_{n+1}^i}, \quad i = 0(1)k$$

$$P_j = \frac{\sum_{i=1}^{j} h^{(j+1-i)} y_n^{(j+1-i)}}{\sum_{i=1}^{j} (j+1-i)! X_{n+1}^{(j+1-i)}} q_{i-1} + y_n q_j, \quad j = 1(1)5.$$

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The results as analyzed with the computer show that the rational interpolation method copes favorably well with ordinary differential equations and stiff initial value problems.

Keywords: Rational interpolation method; Stability; Consistency and convergence

1. Introduction

Numerical methods for systems of Ordinary Differential Equations (ODEs) have been attracting much attention due to their need in the solution of problems arising from the mathematical formulation of physical situations in chemical kinetics, population models, mechanical oscillations, planetary motions, electrical networks, nuclear reactor control, tunnel switching problems, reversible enzyme kinetic often lead to Initial Value Problems (IVPs) in Ordinary Differential Equations that are stiff or singular or oscillatory. There are quite a number that are stiff, they are mainly from Reaction, Chemical kinetic and Life Sciences. (Refs. [1–15].)

The problem for this research paper is to find a numerical solution to the IVP which is represented by,

$$y^1 = f(x, y); \qquad y(x_0) = y_0, \qquad a \le x \le b \tag{1.1}$$

where $f(x, y)$ is defined and continuous in a region $D \subset [a, b]$, that is stiff.

However, this research paper is therefore to design and implement a new method that can cope effectively well with such problems. Our aims in this research paper is to solve problem represented by (1.1) where $f(x, y)$ must satisfy a Lipschitz condition with respect to y. In order to achieve the above aim, we therefore set the following objectives.

(i) To construct a rational interpolation method having one of its members L-stable and to cope with ordinary differential equations and stiff IVPs.
(ii) To determine the performance, stability, consistency and nature of convergence of the rational integrator constructed in (1) above.
(iii) To compare the performance of the proposed rational integrators ($k = 6$) with some established results.
(iv) To develop a program in Java for the implementation of the rational integrators.

Definition 1.1. A one-step method is said to be *A-STABLE* if when applied to the test equation $y^1 = \lambda y$ with $\mathrm{Re}(\lambda) < 0$, it gives,

$y_{n+1} = S\left(\bar{h}\right) y_n$, with the stability function $S\left(\bar{h}\right)$ satisfying,

$$|S(\bar{h})| < 1 \quad \text{for all } \mathrm{Re}\left(\bar{h}\right) < 0,\ \bar{h} = \lambda h.$$

Definition 1.2. A given one-step method is said to be *L-Stable* if it is A-Stable and in addition,

$$\mathrm{Re}\left(\bar{h}\right) \xrightarrow{\text{Limit}} -\infty \left|S\left(\bar{h}\right)\right| = 0.$$

Definition 1.3. The Initial Value Problem (IVP) (1.1.1) is said to be *stiff* over the finite interval $[a, b]$ if for every $x \in [a, b]$, the Eigenvalues $\lambda_i(x)$, $i = 1, 2, \ldots, n$ of the Jacobian matrix arising from (1.1) satisfies the following equations.

1. $\mathrm{Re}\lambda_i(x) < 0$, $i = 1, 2, \ldots, n$ and
2. The stiffness ratio $= \frac{\max\left|\mathrm{Re}\ \lambda_{i(x)}\right|}{\min\left|\mathrm{Re}\ \lambda_{i(x)}\right|} \gg 1$, $i = 1, 2, \ldots, n$.

For the linear initial value problem

$$y^1 = Ay + g, \qquad y(x_0) = y_0 \tag{1.2}$$

$\frac{\partial y^1}{\partial y} = A$, is $m \times m$ Matrix and $g \in R^m$, it is to be noted that A or g may be constraints or functions of x.

2. The derivation of the rational interpolation method

Our derivation of the rational interpolation method consists of matching the Taylor series expansion of $y(x_{n+1})$ with the Taylor series of the approximation value of y_{n+1}. At the point $x = x_n$ in the interval $[x_n, x_{n+1}]$, we set $y_n = y(x_n)$, since Our Taylor series about x_n for both $y(x_{n+1})$ and y_{n+1} require the use of $h = x_{n+1} - x_n$, we choose in sufficiently small enough so that x_{n+1} and x_n are very close.

The following equations are results of matching of the Taylor series of $y(x_{n+1})$ and y_{n+1} for the case $k = 6$ where

$$y_{n+1} = \frac{\sum_{i=0}^{k-1} P_i X_{n+1}^i}{1 + \sum_{i=1}^{k} q_i X_{n+1}^i} = \frac{\sum_{i=0}^{5} P_i X_{n+1}^i}{1 + \sum_{i=1}^{6} q_i X_{n+1}^i} \tag{2.3}$$

$$y_{n+1} = \frac{P_0 + P_1 x_{n+1} + P_2 x_{n+1}^2 + P_3 x_{n+1}^3 + P_4 x_{n+1}^4 + P_5 x_{n+1}^5}{1 + q_1 x_{n+1} + q_2 x_{n+1}^2 + q_3 x_{n+1}^3 + q_4 x_{n+1}^4 + q_5 x_{n+1}^5 + q_6 x_{n+1}^6} \tag{2.4}$$

where,

$$P_j = \sum_{i=1}^{j} \frac{h^{(j+1-i)} y_n^{(j+1-i)} q_{i-1}}{(j+1-i)!\,(x_{n+1})^{(j+1-i)}} + y_n q_j, \quad j = 1(1)k-1 \tag{2.5}$$

when $j = 0,\ i = 1$, from 2.3.3 we have,

$$\begin{aligned} P_0 &= y_n q_0 \quad \text{i.e. } q_0 = 1 \\ P_0 &= y_n \end{aligned} \tag{2.6}$$

when $j = 1,\ i = 1$, (2.5) becomes

$$\begin{aligned} P_1 &= \frac{h y_n^{(1)} q_0}{1! x_{n+1}} + y_n q_1 \Rightarrow \frac{h y_n^{(1)}}{1! x_{n+1}} + y_n q_1 \\ \frac{h y_n^{(1)}}{1!} &= x_{n+1} \left[-p_0 q_1 + p_1 \right] \end{aligned} \tag{2.7}$$

when $j = 2,\ i = 1$, (2.5) becomes

$$\begin{aligned} P_2 &= \frac{h^2 y_n^{(2)}}{2! x_{n+1}^2} + \frac{h y_n^{(1)}}{1! x_{n+1}} q_1 + y_n q_2 \\ \frac{h^2 y_n^{(2)}}{2!} &= x_{n+1}^2 \left[-p_0 q_2 - \frac{h y_n^{(1)}}{1! x_{n+1}} q_1 + p_2 \right] \end{aligned} \tag{2.8}$$

when $j = 3, 4, 5, 6, 7, 8, 9, 10, 11,\ i = 1$ we have in (2.9)

$$\frac{h^3 y_n^{(3)}}{3!} = x_{n+1}^3 \left[-p_0 q_3 - \frac{h^2 y_n^{(2)}}{2! x_{n+1}^2} q_1 - \frac{h y_n^{(1)}}{1! x_{n+1}} q_2 + p_3 \right] \tag{2.9}$$

$$\frac{h^4 y_n^{(4)}}{4!} = x_{n+1}^4 \left[-p_0 q_4 - \frac{h^3 y_n^{(3)}}{3! x_{n+1}^3} q_1 - \frac{h^2 y_n^{(2)}}{2! x_{n+1}^2} q_2 - \frac{h y_n^{(1)}}{1! x_{n+1}} q_3 + p_4 \right] \tag{2.10}$$

$$\frac{h^5 y_n^{(5)}}{5!} = x_{n+1}^5 \left[-p_0 q_5 - \frac{h^4 y_n^{(4)}}{4! x_{n+1}^4} q_1 - \frac{h^3 y_n^{(3)}}{3! x_{n+1}^3} q_2 - \frac{h^2 y_n^{(2)}}{2! x_{n+1}^2} q_3 - \frac{h y_n^{(1)}}{1! x_{n+1}} q_4 + p_5 \right] \tag{2.11}$$

$$\frac{h^6 y_n^{(6)}}{6!} = x_{n+1}^6 \left[-p_0 q_6 - \frac{h^5 y_n^{(5)}}{5! x_{n+1}^5} q_1 - \frac{h^4 y_n^{(4)}}{4! x_{n+1}^4} q_2 - \frac{h^3 y_n^{(3)}}{3! x_{n+1}^3} q_3 - \frac{h^2 y_n^{(2)}}{2! x_{n+1}^2} q_4 - \frac{h y_n^{(1)}}{1! x_{n+1}} q_5 + p_6 \right] \tag{2.12}$$

$$\frac{h^7 y_n^{(7)}}{7!} = x_{n+1}^7\left[-p_0 q_7 - \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6}q_1 - \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}q_2 - \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}q_3 - \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3}q_4 - \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2}q_5 - \frac{h y_n^{(1)}}{1!x_{n+1}}q_6 + p_7\right] \tag{2.13}$$

$$\frac{h^8 y_n^{(8)}}{8!} = x_{n+1}^8\left[-p_0 q_8 - \frac{h^7 y_n^{(7)}}{7!x_{n+1}^7}q_1 - \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6}q_2 - \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}q_3 - \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}q_4 - \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3}q_5 - \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2}q_6 - \frac{h y_n^{(1)}}{1!x_{n+1}}q_7 + p_8\right] \tag{2.14}$$

$$\frac{h^9 y_n^{(9)}}{9!} = x_{n+1}^9\left[-p_0 q_9 - \frac{h^8 y_n^{(8)}}{8!x_{n+1}^8}q_1 - \frac{h^7 y_n^{(7)}}{7!x_{n+1}^7}q_2 - \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6}q_3 - \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}q_4 - \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}q_5 - \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3}q_6 - \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2}q_7 - \frac{h y_n^{(1)}}{1!x_{n+1}}q_8 + p_9\right] \tag{2.15}$$

$$\frac{h^{10} y_n^{(10)}}{10!} = x_{n+1}^{10}\left[-p_0 q_{10} - \frac{h^9 y_n^{(9)}}{9!x_{n+1}^9}q_1 - \frac{h^8 y_n^{(8)}}{8!x_{n+1}^8}q_2 - \frac{h^7 y_n^{(7)}}{7!x_{n+1}^7}q_3 - \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6}q_4 - \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}q_5 - \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}q_6 - \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3}q_7 - \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2}q_8 - \frac{h y_n^{(1)}}{1!x_{n+1}}q_9 + p_{10}\right] \tag{2.16}$$

$$\frac{h^{11} y_n^{(11)}}{11!} = x_{n+1}^{11}\left[-p_0 q_{11} - \frac{h^{10} y_n^{(10)}}{10!x_{n+1}^{10}}q_1 - \frac{h^9 y_n^{(9)}}{9!x_{n+1}^9}q_2 - \frac{h^8 y_n^{(8)}}{8!x_{n+1}^8}q_3 - \frac{h^7 y_n^{(7)}}{7!x_{n+1}^7}q_4 - \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6}q_5 - \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}q_6 - \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}q_7 - \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3}q_8 - \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2}q_9 - \frac{h y_n^{(1)}}{1!x_{n+1}}q_{10} + p_{11}\right] \tag{2.17}$$

$$\vdots$$

$$\frac{h^m y_n^{(m)}}{m!} = x_{n+1}^m\left[-y_n q_i - \frac{h^{m-v} y_n^{(m-v)}}{(m-v)!x_{n+1}^{m-v}}q_{i-(m-1)} - \frac{h^{m-v+1} y_n^{(m-v+1)}}{(m-v+1)!x_{n+1}^{m-v+1}}q_{i-(m-2)} - \frac{h^{m-v+2} y_n^{(m-v+2)}}{(m-v+2)!x_{n+1}^{m-v+2}}q_{i-(m-3)} - \frac{h^{mv+3} y_n^{(m-v+3)}}{(m-v+3)!x_{n+1}^{m-v+3}}q_{i-(m-4)} - \frac{h^{m-v+4} y_n^{(m-v+4)}}{(m-v+4)!x_{n+1}^{m-v+4}}q_{i-(m-5)} - \frac{h^{m-v+5} y_n^{(m-v+5)}}{(m-v+5)!x_{n+1}^{m-v+5}}q_{i-(m-6)} - \frac{h^{m-v+6} y_n^{(m-v+6)}}{(m-v+6)!x_{n+1}^{m-v+6}}q_{i-(m-7)} - \cdots - \frac{h^{m-v+r} y_n^{(m-v+r)}}{(m-v+r)!x_{n+1}^{m-v+r}}q_{i-(m-t)} + p_i\right]. \tag{2.18}$$

Hence Eq. (2.18) gives our general derivation method. And where $P_0 = y_n,\ v = 1,\ m = i = 1, 2, 3, \ldots,\ t = 1, 2, 3, \ldots$ and $r = 1, 2, 3, \ldots$. As the parameters of $p_1, p_2, p_3, p_4, p_5, q_1, q_2, q_3, q_4, q_5$ and q_6 cannot be easily obtained using Eqs. (2.8)–(2.17). We therefore adopt the method of writing one result as a combination of others as was used by Fatunla [16]. By writing (2.17) as a combination of Eqs. (2.8)–(2.16) we get,

$$\frac{h^{10} y_n^{(10)}}{10!x_{n+1}^{10}}q_1 + \frac{h^9 y_n^{(9)}}{9!x_{n+1}^9}q_2 + \frac{h^8 y_n^{(8)}}{8!x_{n+1}^8}q_3 + \frac{h^7 y_n^{(7)}}{7!x_{n+1}^7}q_4 + \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6}q_5 + \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}q_6 = -\frac{h^{11} y_n^{(11)}}{11!x_{n+1}^{11}} \tag{2.19}$$

$$\frac{h^9 y_n^{(9)}}{9!x_{n+1}^9}q_1 + \frac{h^8 y_n^{(8)}}{8!x_{n+1}^8}q_2 + \frac{h^7 y_n^{(7)}}{7!x_{n+1}^7}q_3 + \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6}q_4 + \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}q_5 + \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}q_6 = -\frac{h^{10} y_n^{(10)}}{10!x_{n+1}^{10}} \tag{2.20}$$

$$\frac{h^8 y_n^{(8)}}{8!x_{n+1}^8}q_1 + \frac{h^7 y_n^{(7)}}{7!x_{n+1}^7}q_2 + \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6}q_3 + \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}q_4 + \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}q_5 + \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3}q_6 = -\frac{h^9 y_n^{(9)}}{9!x_{n+1}^9} \tag{2.21}$$

$$\frac{h^7 y_n^{(7)}}{7!x_{n+1}^7}q_1 + \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6}q_2 + \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}q_3 + \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}q_4 + \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3}q_5 + \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2}q_6 = -\frac{h^8 y_n^{(8)}}{8!x_{n+1}^8} \tag{2.22}$$

$$\frac{h^6 y_n^{(6)}}{6!x_{n+1}^6}q_1 + \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}q_2 + \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}q_3 + \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3}q_4 + \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2}q_5 + \frac{h y_n^{(1)}}{1!x_{n+1}}q_6 = -\frac{h^7 y_n^{(7)}}{7!x_{n+1}^7} \tag{2.23}$$

$$\frac{h^5 y_n^{(5)}}{5!x_{n+1}^5}q_1 + \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}q_2 + \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3}q_3 + \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2}q_4 + \frac{h y_n^{(1)}}{1!x_{n+1}}q_5 + y_n q_6 = -\frac{h^6 y_n^{(6)}}{6!x_{n+1}^6}. \tag{2.24}$$

For each positive integer m, the expression:

$$\frac{h^m y_n^m}{m!x_{n+1}^m}$$

is a real number, for this reason Eqs. (2.19)–(2.24) represent simultaneous Linear Equation (SLE) in q_1, q_2, q_3, q_4, q_5 and q_6. In the course of using the integrator (2.3) of higher values of k, the use of matrix equation gives rise to clearer solutions and make clearer the investigation of stability properties of the integrator, hence we put the simultaneous linear equation in matrix form as shown below:

$$\begin{bmatrix}
\frac{h^{10} y_n^{(10)}}{10!x_{n+1}^{10}} & \frac{h^0 y_n^{(0)}}{9!x_{n+1}^9} & \frac{h^8 y_n^{(8)}}{8!x_{n+1}^8} & \frac{h^7 y_n^{(7)}}{7!x_{n+1}^7} & \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6} & \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5} \\
\frac{h^9 y_n^{(9)}}{9!x_{n+1}^9} & \frac{h^8 y_n^{(8)}}{8!x_{n+1}^8} & \frac{h^7 y_n^{(7)}}{7!x_{n+1}^7} & \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6} & \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5} & \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4} \\
\frac{h^8 y_n^{(8)}}{8!x_{n+1}^8} & \frac{h^7 y_n^{(7)}}{7!x_{n+1}^7} & \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6} & \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5} & \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4} & \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3} \\
\frac{h^7 y_n^{(7)}}{7!x_{n+1}^7} & \frac{h^6 y_n^{(6)}}{6!x_{n+1}^6} & \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5} & \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4} & \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3} & \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2} \\
\frac{h^6 y_n^{(6)}}{6!x_{n+1}^6} & \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5} & \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4} & \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3} & \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2} & \frac{h y_n^{(1)}}{1!x_{n+1}} \\
\frac{h^5 y_n^{(5)}}{5!x_{n+1}^5} & \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4} & \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3} & \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2} & \frac{h y_n^{(1)}}{1!x_{n+1}} & y_n
\end{bmatrix}
\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}
=
\begin{bmatrix}
-\frac{h^{11} y_n^{(11)}}{11!x_{n+1}^{11}} \\
-\frac{h^{10} y_n^{(10)}}{10!x_{n+1}^{10}} \\
-\frac{h^9 y_n^{(9)}}{9!x_{n+1}^9} \\
-\frac{h^8 y_n^{(8)}}{8!x_{n+1}^8} \\
-\frac{h^7 y_n^{(7)}}{7!x_{n+1}^7} \\
-\frac{h^6 y_n^{(6)}}{6!x_{n+1}^6}
\end{bmatrix} = b. \tag{2.25}$$

To solve for q_1, q_2, q_3, q_4, q_5 and q_6 we use the matrix equation (2.25) p_1, p_2, p_3, p_4, and p_5 can be obtained from the Eqs. (2.6)–(2.11). That is,

$$P_1 = \frac{h y_n^{(1)} q_0}{1!x_{n+1}} + y_n q_1 \tag{2.26}$$

$$P_2 = \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2} + \frac{h y_n^{(1)}}{1!x_{n+1}}q_1 + y_n q_2 \tag{2.27}$$

$$P_3 = \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3} + \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2}q_1 + \frac{h y_n^{(1)}}{1!x_{n+1}}q_2 + y_n q_3 \tag{2.28}$$

$$P_4 = \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4} + \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3}q_1 + \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2}q_2 + \frac{h y_n^{(1)}}{1!x_{n+1}}q_3 + y_n q_4 \tag{2.29}$$

$$P_5 = \frac{h^5 y_n^{(5)}}{5!x_{n+1}^5} + \frac{h^4 y_n^{(4)}}{4!x_{n+1}^4}q_1 + \frac{h^3 y_n^{(3)}}{3!x_{n+1}^3}q_2 + \frac{h^2 y_n^{(2)}}{2!x_{n+1}^2}q_3 + \frac{h y_n^{(1)}}{1!x_{n+1}}q_4 + y_n q_5. \tag{2.30}$$

By adopting Eqs. (2.26)–(2.30) produces the values of p_1, p_2, p_3, p_4 and p_5 when the values of q_1, q_2, q_3, q_4 and q_5 are substituted into it.

3. Stability of the rational interpolation method

In this section, we shall investigate the stability properties of the integrator formula for $k = 6$.

$$y_{n+1} = \frac{\sum_{i=0}^{5} P_i X_{n+1}^i}{1 + \sum_{i=1}^{6} q_i X_{n+1}^i}. \tag{3.31}$$

However, the use of matrix equation (2.25) will give us investigative advantage to obtain results easily. When the integrator (3.31) is applied to $y^1 = \lambda y$ the Eq. (2.25) becomes:

$$\begin{bmatrix} \frac{\overline{h}^{10}}{10!x_{n+1}^{10}} & \frac{\overline{h}^{9}}{9!x_{n+1}^{9}} & \frac{\overline{h}^{8}}{8!x_{n+1}^{8}} & \frac{\overline{h}^{7}}{7!x_{n+1}^{7}} & \frac{\overline{h}^{6}}{6!x_{n+1}^{6}} & \frac{\overline{h}^{5}}{5!x_{n+1}^{5}} \\ \frac{\overline{h}^{9}}{9!x_{n+1}^{9}} & \frac{\overline{h}^{8}}{8!x_{n+1}^{8}} & \frac{\overline{h}^{7}}{7!x_{n+1}^{7}} & \frac{\overline{h}^{6}}{6!x_{n+1}^{6}} & \frac{\overline{h}^{5}}{5!x_{n+1}^{5}} & \frac{\overline{h}^{4}}{4!x_{n+1}^{4}} \\ \frac{\overline{h}^{8}}{8!x_{n+1}^{8}} & \frac{\overline{h}^{7}}{7!x_{n+1}^{7}} & \frac{\overline{h}^{6}}{6!x_{n+1}^{6}} & \frac{\overline{h}^{5}}{5!x_{n+1}^{5}} & \frac{\overline{h}^{4}}{4!x_{n+1}^{4}} & \frac{\overline{h}^{3}}{3!x_{n+1}^{3}} \\ \frac{\overline{h}^{7}}{7!x_{n+1}^{7}} & \frac{\overline{h}^{6}}{6!x_{n+1}^{6}} & \frac{\overline{h}^{5}}{5!x_{n+1}^{5}} & \frac{\overline{h}^{4}}{4!x_{n+1}^{4}} & \frac{\overline{h}^{3}}{3!x_{n+1}^{3}} & \frac{\overline{h}^{2}}{2!x_{n+1}^{2}} \\ \frac{\overline{h}^{6}}{6!x_{n+1}^{6}} & \frac{\overline{h}^{5}}{5!x_{n+1}^{5}} & \frac{\overline{h}^{4}}{4!x_{n+1}^{4}} & \frac{\overline{h}^{3}}{3!x_{n+1}^{3}} & \frac{\overline{h}^{2}}{2!x_{n+1}^{2}} & \frac{\overline{h}}{1!x_{n+1}} \\ \frac{\overline{h}^{5}}{5!x_{n+1}^{5}} & \frac{\overline{h}^{4}}{4!x_{n+1}^{4}} & \frac{\overline{h}^{3}}{3!x_{n+1}^{3}} & \frac{\overline{h}^{2}}{2!x_{n+1}^{2}} & \frac{\overline{h}}{1!x_{n+1}} & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} \frac{\overline{h}^{11}}{11!x_{n+1}^{11}} \\ \frac{\overline{h}^{10}}{10!x_{n+1}^{10}} \\ \frac{\overline{h}^{9}}{9!x_{n+1}^{9}} \\ \frac{\overline{h}^{8}}{8!x_{n+1}^{8}} \\ \frac{\overline{h}^{7}}{7!x_{n+1}^{7}} \\ \frac{\overline{h}^{6}}{6!x_{n+1}^{6}} \end{bmatrix} = b \tag{3.32}$$

where $\overline{h} = \lambda h$.

The application of Gaussian elimination to (3.32) yields

$$\begin{bmatrix} 1 & \frac{10X_{n+1}}{\overline{h}} & \frac{90X_{n+1}^2}{\overline{h}^2} & \frac{720X_{n+1}^3}{\overline{h}^3} & \frac{5040X_{n+1}^4}{\overline{h}^4} & \frac{302\,405X_{n+1}^5}{\overline{h}^5} \\ 0 & 1 & \frac{18X_{n+1}}{\overline{h}} & \frac{216X_{n+1}^2}{\overline{h}^2} & \frac{5016X_{n+1}^3}{\overline{h}^3} & \frac{15\,120X_{n+1}^4}{\overline{h}^4} \\ 0 & 0 & 1 & \frac{24X_{n+1}}{\overline{h}} & \frac{336X_{n+1}^2}{\overline{h}^2} & \frac{3360X_{n+1}^3}{\overline{h}^3} \\ 0 & 0 & 0 & 1 & \frac{28X_{n+1}}{\overline{h}} & \frac{420X_{n+1}^2}{\overline{h}^2} \\ 0 & 0 & 0 & 0 & 1 & \frac{30X_{n+1}}{\overline{h}} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} -\frac{10!\overline{h}}{11!X_{n+1}} \\ -\frac{9!\overline{h}^2}{11!X_{n+1}^2} \\ -\frac{8!\overline{h}^3}{11!X_{n+1}^3} \\ -\frac{7!\overline{h}^4}{11!X_{n+1}^4} \\ -\frac{6!\overline{h}^5}{11!X_{n+1}^5} \\ -\frac{5!\overline{h}^6}{11!X_{n+1}^6} \end{bmatrix} \tag{3.33}$$

where

$$\begin{bmatrix} q_1\left(\overline{h}\right) \\ q_2\left(\overline{h}\right) \\ q_3\left(\overline{h}\right) \\ q_4\left(\overline{h}\right) \\ q_5\left(\overline{h}\right) \\ q_6\left(\overline{h}\right) \end{bmatrix} = \begin{bmatrix} -\dfrac{3\,628\,200\overline{h}}{11!X_{n+1}} \\ -\dfrac{13\,426\,560\overline{h}^2}{11!X_{n+1}^2} \\ -\dfrac{2\,177\,280\overline{h}^3}{11!X_{n+1}^3} \\ -\dfrac{166\,320\overline{h}^4}{11!X_{n+1}^4} \\ -\dfrac{4320\overline{h}^5}{11!X_{n+1}^5} \\ -\dfrac{120\overline{h}^6}{11!X_{n+1}^6} \end{bmatrix}. \tag{3.34}$$

Applying (3.34) in (2.26)–(2.30) also imply

$$\begin{bmatrix} p_1\left(\overline{h}\right) \\ p_2\left(\overline{h}\right) \\ p_3\left(\overline{h}\right) \\ p_4\left(\overline{h}\right) \\ p_5\left(\overline{h}\right) \end{bmatrix} = \begin{bmatrix} \dfrac{36\,288\,600\overline{h}}{11!X_{n+1}} \\ \dfrac{29\,756\,760\overline{h}^2}{11!X_{n+1}^2} \\ \dfrac{16\,088\,060\overline{h}^3}{11!X_{n+1}^3} \\ \dfrac{5\,670\,820\overline{h}^4}{11!X_{n+1}^4} \\ \dfrac{1\,492\,585\overline{h}^5}{11!X_{n+1}^5} \end{bmatrix}. \tag{3.35}$$

Finally, if we apply the results in (3.34) and (3.35) to the integrator (3.31). We obtain the stability root given by

$$S\left(\overline{h}\right) = \frac{P_0 + P_1x_{n+1} + P_2x_{n+1}^2 + P_3x_{n+1}^3 + P_4x_{n+1}^4 + P_5x_{n+1}^5}{1 + q_1x_{n+1} + q_2x_{n+1}^2 + q_3x_{n+1}^3 + q_4x_{n+1}^4 + q_5x_{n+1}^5 + q_6x_{n+1}^6}$$

$$S\left(\overline{h}\right) = \frac{3\,991\,680 + 36\,288\,600\overline{h} + 29\,756\,760\overline{h}^2 + 1608\,806\overline{h}^3 + 567\,020\overline{h}^4 + 1\,492\,585\overline{h}^5}{3\,991\,680 - 3\,628\,200\overline{h} + 13\,426\,560\overline{h}^2 - 2\,177\,280\overline{h}^3 + 166\,320\overline{h}^4 - 4320\overline{h}^5 + 120\overline{h}^6}. \tag{3.36}$$

However, by direct proof and definitions, Eq. (3.36) is A-stable and L-stable. For L-stable the RAS together with the point of encroachment into the right-half of the complex plane.

4. Consistency and convergence

The integration formula given by $y_{n+1} = y_n + h\phi(x_n, y_n, h)$ is said to be consistent with the initial value problem (ivp), if

$$\phi(x, y, 0) = f(x, y). \tag{4.37}$$

With this definition, we can prove the theorem on the consistency of our method.

Lambert [17] further stated that a multi step is said to be convergent if and only if the method is zero stable and consistent.

Theorem (*Consistency and Convergence*). *The numerical integration formula given by,*

$$y_{n+1} = \frac{\sum_{i=0}^{k-1} P_i X_{n+1}^i}{1 + \sum_{i=1}^{k} q_i X_{n+1}^i} \tag{4.38}$$

is convergent.

Proof. To establish the convergence of the new integrators formula, we have to show that the integrators are consistent as noted in [18].

Given that $P_0 = y_n$ (By (2.6))

$$y_{n+1} = \frac{\sum_{i=0}^{5} P_i X_{n+1}^i}{1 + \sum_{i=1}^{6} q_i X_{n+1}^i} \quad \text{(By (2.3))}.$$

By re-arranging (2.3) we obtain

$$y_{n+1} - y_n = \frac{\sum_{i=0}^{5} (P_i - q_i y_n) X_{n+1}^i}{1 + \sum_{i=1}^{6} q_i X_{n+1}^i} \tag{4.39}$$

$$\frac{y_{n+1} - y_n}{h} = \frac{(p_1 - q_1 y_n)_{n+1} + (p_2 - q_2 y_n)(n+1)^2 h + (p_3 - q_3 y_n)(n+1)^3 h^2 + (p_4 - q_4 y_n)(n+1)^4 h^3 - q_5 (n+1)^5 h^4}{1 + \sum_{i=1}^{6} q_i (n+1)^i h^i} \tag{4.40}$$

$$\frac{y_{n+1} - y_n}{h} = \phi(x_n, y_n, h) \tag{4.41}$$

where ϕ is the increment function.

This leads us to,

$$\phi(x_n, y_n, 0) = (p_1 - q_1 y_n)(n+1) \quad \text{i.e. } n+1 = \frac{x_{n+1}}{h}$$

$$\phi(x_n, y_n, 0) = (p_1 - q_1 y_n)\frac{x_{n+1}}{h}. \tag{4.42}$$

But,

$$p_i = \sum_{j=1}^{i} \frac{h^{i+1-j} (y_n)^{i+1-j} q_{j-1}}{(i+1-j)! (x_{n+1})^{i+1-j}} + y_n q_j \tag{4.43}$$

$$p_1 = \frac{h y_n^{(1)}}{1! x_{n+1}} + y_n q_1$$

$$p_1 - y_n q_1 = \frac{h y_n^{(1)}}{1! x_{n+1}} \tag{4.44}$$

putting Eq. (4.44) into (4.42)

$$\phi(x_n, y_n, 0) = \frac{h y_n^{(1)}}{x_{n+1}} \cdot \frac{x_{n+1}}{h}$$

Table 1
Efficiency of our method of order eleven (11) on Problem 1 for $h = 0.5$, and $0 \le x \le 5$.

x_n	H	Theoretical solution (Tsol)	Numerical solution (Nsol)	Interpolation method error (En)	N_f
0.5	0.5	1.648721271	1.648721271	−1.386823989E−10	1
1.0	0.5	2.718281828	2.718281826	1.801506944E−9	2
1.5	0.5	4.481689069	4.481688764	3.047049768E−7	3
2.0	0.5	7.389056096	7.389046016	1.008072234E−5	4
2.5	0.5	12.182493956	12.182340480	1.534755087E−4	5
3.0	0.5	20.085536913	20.084103084	1.433828596E−3	6
3.5	0.5	33.115451939	33.105881851	9.570088402E−3	7
4.0	0.5	54.598149996	54.548180215	4.996978142E−2	8
4.5	0.5	90.017131232	89.800704590	2.164266417E−1	9
5.0	0.5	148.413158977	148.603848505	8.093104724E−1	10

Table 2
Error in numerical integrators with uniform meshsize $h = 0.1$ on Problem 1.

X	Theoretical solution	Aashikpelokhai [1]				Fatunla (1983)	N_f
		Error $\times 10^{12}$ Order = 11	Error $\times 10^{12}$ Order = 9	Error $\times 10^{12}$ Order = 7	Error $\times 10^{8}$ Order = 5	Error $\times 10^{6}$ Order = 4	
0.1	1.10517092	0.00000	0.00000	0.00799	−0.15622	0.60450	1
0.2	1.22140027	0.00022	−0.00022	0.00866	−0.17265	0.48269	2
0.3	1.34985881	0.00000	0.00000	0.00955	−0.19081	0.38781	3
0.4	1.49182470	−0.00022	0.00000	0.01066	−0.21088	0.36697	4
0.5	1.64872127	0.00022	0.00000	0.01177	−0.23305	0.32105	5
0.6	1.82211880	−0.00022	−0.00022	0.01288	−0.25756	0.02256	6
0.7	2.01375271	0.00000	0.00000	0.01465	−0.28465	0.35937	7
0.8	2.22554093	−0.00044	0.00000	0.01554	−0.31459	0.34473	8
0.9	2.45960311	0.00000	0.00000	0.01731	−0.34768	0.30697	9
1.0	2.71828183	0.00044	0.00000	0.01954	−0.38424	0.03030	10

$$\phi(x_n, y_n, 0) = y_n^{(1)} = f(x_n, y_n) = f(x, y).$$

Hence, we conclude that the new rational interpolation method is consistent with the initial value problem (ivp).

5. Numerical computations and results

Problem 1 (*Aashikpelokhai [1] and Fatunla (1983)*)**.**

$$y^1 = y, \qquad y(0) = 1, \qquad \text{Exact Solution } y = e^x, \quad 0 \le x \le 1.$$

From Table 1, we observed that our new rational interpolation method of order eleven when compared with Table 2 above shows that the rational interpolation method are efficient at resolving differential equations with exponential solutions. They compare favorably with Aashikpelokhai [1] and Fatunla (1983).

Problem 2 (*Corless [7]*)**.**

$$y' = x - y, \qquad y(1) = 2, \qquad \text{Exact Solution } y = x - 1 + 2e^{1-x}$$
$$1.0 \le x \le 2.0 \quad \text{and} \quad h = 0.1.$$

From the numerical results in Table 3, we observed that the error tolerance of order eleven could raise to 10^{-11} as against of order five 10^{-04} prescribed in [7]. Hence order eleven of our new interpolation method has a higher degree of accuracy.

Table 3

X	H	Theoretical solution (Tsol)	Corless error Order = 5	Interpolation method error Order = 11	N_f
1.1	0.1	1.90967483610	2.951622E−04	3.056044E−11	11
1.2	0.1	1.83746150621	5.341768E−04	5.530487E−11	12
1.3	0.1	1.78163644144	7.249117E−04	7.506484E−11	13
1.4	0.1	1.74064009216	8.745193E−04	9.062775E−11	14
1.5	0.1	1.71306131953	9.890795E−04	1.034086E−10	15
1.6	0.1	1.69762327230	1.073837E−03	1.199023E−10	16
1.7	0.1	1.69317060770	1.133561E−03	1.722329E−10	17
1.8	0.1	1.69865792836	1.172066E−03	3.916361E−10	18
1.9	0.1	1.71313931960	1.193047E−03	1.226139E−9	19
2.0	0.1	1.73575888247	1.199245E−03	3.999923E−9	20

Table 4

X_n	H	Theoretical solution	Aashikpelokhai [1]		Interpolation method	N_f
			Error $K = 2$	Error $K = 1$	Error $k = 6$	
First component						
5.0	0.2	0.606530660	−1.3(−9)	−1.2(−4)	−2.4E−11	25
10.0	0.2	0.367879441	−8.1(−10)	−7.3(−5)	−3.8E−11	50
15.0	0.2	0.223313016	−4.9(−10)	−4.4(−5)	−4.3E−11	75
Second component						
5.0	0.2	0.000000000	0.0	0.0	0.0	25
10.0	0.2	0.000000000	0.0	0.0	0.0	50
15.0	0.2	0.000000000	0.0	0.0	0.0	75
Third component						
5.0	0.2	0.000000000	0.0	0.0	0.0	25
10.0	0.2	0.000000000	0.0	0.0	0.0	50
15.0	0.2	0.000000000	0.0	0.0	0.0	75

Problem 3.

$$y' = \begin{bmatrix} -0.1 & -49 & 0 \\ 0 & -50 & 0 \\ 0 & 70 & -120 \end{bmatrix} y, \qquad y(0) = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad 0 \le x \le 15, \qquad \overset{\text{Exact solution}}{y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-0.1x} \\ e^{-50x} \\ e^{-120x} \end{bmatrix}}.$$

The eigenvalues of the system of the differential equation are $\lambda_1 = -0.1$, $\lambda_2 = -50$ and $\lambda_3 = -120$. Therefore the stiffness factor is 1200.

From Table 4 above we can observe that our new rational interpolation method also shows the accuracies of the integrators with increasing in k.

6. Conclusion

The implementations of the class of rational interpolation method were coded in Java and run on digital computer.

In conclusion, from the above results in Tables 1–4 it can be seen that our Rational Interpolation method of order eleven is efficient and accurate when compared with the existing methods of Aashikpelokhai [1] and Fatunla (1983) and Corless [7] which can solve the same set of numerical initial value problems. The Region of Absolute Stability (RAS) of the rational Interpolation method lies entirely on the Left-half of the complex plane. We therefore conclude that the rational interpolation method is both A-stable and L-stable for $K = 6$. Hence, it is recommended for users, who are currently working in the area of research.

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Further reading

[1] Lambert JD, Shaw B. On the numerical solution of $y' = f(x, y)$ by a class of formulae based on rational approximation. Math Comp 1965;19:456–62.